Finite Math - Spring 2017

Lecture Notes - 2/27/2017

Homework

- Section 3.4 - 44, 49, 50

Section 3.4 - Present Value of an Annuity; Amortization

An interesting application of present value in conjunction with sinking funds is saving for retirement.

Example 1. *The full retirement age in the US is* 67 *for people born in* 1960 *or later. Suppose you start saving for retirement at* 27 *years old and you would like to save enough to withdraw* \$40,000 *per year for the next* 20 *years. If you find a retirement savings account (for example, a Roth IRA) which pays* 4% *interest compounded annually, how much will you have to deposit per year from age* 27 *until you retire in order to be able to make your desired withdraws?*

Solution. *First, we should figure out how much money we need to have in the account at the time we retire in order to be able to make the withdraws each year. For this situation, we have*

$$PMT = \$40,000, \quad m = 1, \quad r = 0.04, \quad t = 20, \quad n = mt = 20$$

and so the present value of the retirement account at the time of retirement needs to be

$$PV = \$40,000\frac{1-\left(1+\frac{0.04}{1}\right)^{-20}}{0.04/1} = \$543,613.05.$$

So, now that we know how much we need to have in the account at the time of retirement, we can figure out how much we need to deposit into the savings account per year in order to achieve that amount in the 40 *years we have to save. To do this, we use the sinking fund formula. The future value here is the value we want at the time of retirement, so*

$$FV = \$543,613.05$$

and the other numbers are

$$r = 0.04, \quad m = 1, \quad t = 40, \quad n = mt = 40$$

Plugging this in the formula gives

$$PMT = \$543,613.05\frac{0.04/1}{\left(1+\frac{0.04}{1}\right)^{40}-1} = \$5,720.71.$$

So we will have to deposit \$5,720.71 *per year from age* 27 *until retirement into this account in order to be able to withdraw* \$40,000 *per year for* 20 *years.*

Example 2. *Lincoln Benefit Life offered an ordinary annuity earning* 6.5% *compounded annually. If* $\$2,000$ *is deposited annually for the first* 25 *years, how much can be withdrawn annually for the next* 20 *years?*

Solution. $\$10,688.87$

Amortization. *Amortization* is the process of paying off a debt. The formula for present value of an annuity will allow us to model the process of paying off a loan or other debt. The reason the formula is the same is because receiving payments from your savings account is essentially the bank repaying you the money you loaned them by depositing it into a savings account.

Example 3. *Suppose you take out a* 5*-year,* $\$25,000$ *loan from your bank to purchase a new car. If your bank gives you* 1.9% *interest compounded monthly on the loan and you make equal monthly payments, how much will your monthly payment be?*

Solution. *Since the loan was* $\$25,000$ *and it is being paid off, the present value will be* $PV = \$25,000$*. The interest rate is* $r = 0.019$ *and is compounded monthly,* $m = 12$*. The loan lasts for* 5 *years, so we get*

$$\$25,000 = PMT\frac{1-\left(1+\frac{0.019}{12}\right)^{-60}}{\frac{0.019}{12}} = 57.19500PMT$$

Thus, solving for PMT *gives*

$$PMT = \frac{\$25,000}{57.19500} = \$437.10$$

which means our monthly payment would be $\$437.10$ *for* 5 *years.*

We get the following formula

Definition 1 (Amortization)**.**

$$PMT = PV\frac{i}{1-(1+i)^{-n}}$$

where all the variables have the same meaning as for annuities.

Example 4. *If you sell your car to someone for* $\$2,400$ *and agree to finance it at* 1% *per month on the unpaid balance, how much should you receive each month to amortize the loan in* 24 *months? How much interest will you receive?*

Solution. $PMT = \$112.98$*,* $I = \$311.52$

Amortization Schedules. Suppose you are amortizing a debt by making equal payments, but then decided to pay off the debt with one lump-sum payment. How do you find the “pay-off” balance of the debt? (E.g., you take out a 5-year loan with monthly payments for a car, but after 3-years of making payments you decide to just make one final payment to retire the debt.) This "pay-off" is very useful, even if you are not retiring the debt, but refinancing it. When refinancing a debt, you are essentially taking out a new loan to pay-off the previous debt, so you need to know how much unpaid balance remains on the account. When you are making payments into an amortization, at the beginning, a large part of your payment goes towards interest, while later, a larger part goes towards the unpaid balance.

We can see how much of each payment goes towards interest and how much towards unpaid balance by creating an *amortization schedule*.

Example 5. *Construct the amortization schedule for a* $1,000 *debt that is to be amortized in six equal monthly payments at* 1.25% *interest per month on the unpaid balance.*

Solution. *The first step in this process is to compute the required monthly payment using the amortization formula*

$$PMT = \$1,000\frac{0.0125}{1-(1+0.0125)^{-6}} = \$174.03$$

Now, to figure out how much of the payment goes towards interest and how much towards unpaid balance, we compute the interest due at the end of the first month:

$$\$1,000(0.0125) = \$12.50$$

and so the amount of the payment that goes towards the unpaid balance is:

$$\$174.03 - \$12.50 = \$161.53.$$

Thus, the unpaid balance at the end of the first month is

$$\$1,000 - \$161.53 = \$838.47.$$

To compute the breakdown for the next month, we do the same thing, but with the new unpaid balance. The interest due at the end of month 2*:*

$$\$838.47(0.0125) = \$10.48$$

amount of payment towards unpaid balance:

$$\$174.03 - \$10.48 = \$163.55$$

and so the unpaid balance at the end of 2 *months is*

$$\$838.47 - \$163.55 = \$674.92.$$

Payment Number	**Payment**	**Interest**	**Unpaid Balance Reduction**	**Unpaid Balance**
0				$1,000
1	$174.03	$12.50	$161.53	$838.47
2	$174.03	$10.48	$163.55	$674.92
3	$174.03	$8.44	$165.59	$509.33
4	$174.03	$6.37	$167.66	$341.67
5	$174.03	$4.27	$169.76	$171.91
6	$174.06	$2.15	$171.91	$0.00
Total	$1,044.21	$44.21	$1,000	